СЕКЦІЯ 1.
ТЕХНОЛОГІЧНІ ПРОЦЕСИ ТА СИСТЕМИ МАШИНОБУДІВНОГО ВИРОБНИЦТВА

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Stephen Radzevich, Dr.Sci, Professor
EATON Corp., USA, radzevich@usa.com

ON THE EVOLUTION OF CONJUGATE ACTION LAW (IN SCIENTIFIC THEORY OF GEARING)

This research deals with gearing, with crossed-axes gearing (C_a – *gearing*) in particular.

The *scientific theory of gearing* is based on three fundamental laws of gearing, namely, on [1]:

1. ***The law of contact*** *of tooth flanks of a gear,* $\mathscr{G}$*, and of a mating pinion,* $\mathscr{P}$*.* This, in nature, is the *first fundamental law of gearing*.
2. ***The conjugate action law***, which is commonly referred to the *second fundamental law of gearing*.
3. ***The law of equal base pitches*** *in gearing*. This is the *third fundamental law of gearing* that generalizes the first two laws (clauses 1 and 2) [1].

In the rest of text that immediately follows, evolution of the *conjugate action law* is discussed in more detail.

For a simplified case of *parallel-axes gearing* (P_a – *gearing*), the conjugate action law has been formulated, and has been proven in the mid of 18th century. Three names are credited with the discovery of the conjugate action law. They are (in a chronological order):

- *Charles Etienne Louis* ***Camus*** (1699–1768) – a French mathematician and mechanician;
- *Leonhard* ***Euler*** (1707 – 1783) – a pioneering Swiss mathematician and physicist;
- *Felix* ***Savary*** (1797 – 1841) – a French mathematician and mechanician.

Lots of efforts were undertaken by *Camus* to formulate, and to prove the conjugate action law for P_a – *gearing*. Despite *Camus* failed to solve this challenging problem, he made significant contribution to the theory of gearing.

The problem under consideration was solved by *Euler*. The derived solution to the problem was published in two famous papers by *Euler* [2] and [3]. In about 50 years later, that same problem was independently of *Euler* solved by *Savary*. The representation of the results of the research obtained by *Savary*, is very close to that used in the nowadays theory of gearing.

Taken in a whole the contribution by *Camus*, *Euler*, and *Savary*, to the developments in the field of conjugate action law, the second fundamental law of P_a – *gearing* is referred to the *Camus-Euler-Savary fundamental law of parallel-axes gearing* (of just CES – *fundamental law of* P_a – *gearing*, for simplicity).

The CES – *fundamental law of parallel-axes gearing* is known in European countries mainly due to the 1841 book by *Robert Willis* [4]:

Fundamental theorem of parallel-axes gearing (according to *R. Willis* [4]**):** *The angular velocities of the two pieces are to each other inversely as the segments into which the "line of action" divides the line of centers, or inversely as the perpendiculars from centers of motion upon the line of action*.

This theorem is loosely called *Willis' theorem of parallel-axes gearing*, which is incorrect, as no contribution by *Willis* to the CES – *fundamental law of parallel-axes gearing* is known. Moreover, *Willis* himself never claimed to be the author of this theorem.

After the CES – *fundamental law of parallel-axes gearing* was formulated, and has been proven, an evolution of this law of gearing to the case of intersected-axes gearing (I_a – *gearing*), and to the crossed-axes gearing (C_a – *gearing*), becomes natural.

Two gear researchers deserve to be credited for the discovery of *conjugate action law* for I_a *– gearing*. The first valuable portion of the research in this area has been undertaken by *George Barnard* ***Grant*** (1849 – 1917) in his USA patent of 1887 [5]. *Grant* has investigated rolling with no slippage of a cone of revolution over a plane. The geometry of tooth flank of geometrically-accurate bevel gear was derived on the premise of this investigation. However, the task was not been complete, as the rolling of only one cone of revolution was investigated. [*Grant* is considered one of the founders of gear-cutting industry in the USA]. The problem under consideration got a complete solution (~2008) by Prof. *S.P. Radzevich*, who investigated a timed rotation with no slippage of two cones of revolution over a plane. Rolling of two cones of revolution was interpreted by *Radzevich* as a kind of "*equivalent pulley-and-belt transmission*".

A solution to the problem of *conjugate action law* for C_a *– gearing* was resolved (~2008) by Prof. *S.P. Radzevich*. The derived solution is based on the proposed "*equivalent pulley-and-belt transmission*" [1], that was evolved to the case of C_a *– gearing*.

In the most general case of *crossed-axes gearing*, the *second fundamental law of gearing* is formulated as follows:

The second fundamental law of gearing (general case): *In order to smoothly transmit a uniform rotary motion from a driving shaft to a driven shaft by means of gear teeth, the perpendicular to the tooth flanks of the interacting teeth at all points of their contact must intersect the axis of instant rotation in the gear pair*.

The *second fundamental law of gearing* is analytically represented as a set of two expressions:

$$\begin{cases} \boldsymbol{\omega}_{pl} \times \mathbf{V}_m \cdot \mathbf{n}_g = 0 \\ \mathbf{p}_{ln} \times \mathbf{n}_g \neq 0 \end{cases} \tag{1}$$

Here, in Eq. (1), is designated:

$\boldsymbol{\omega}_{pl}$ – is the vector of instant rotation in the C_a gear pair

$\mathbf{p}_{ln}$ – is the unit vector along the axis of instant rotation, P_{ln}

$\mathbf{V}_m$ – is the linear velocity vector of point of interest, m, taken within the instant line of contact, LC_{inst}, between the interacting tooth flanks, $\mathscr{G}$ and $\mathscr{P}$, of a gear and a mating pinion

$\mathbf{n}_g$ – is the unit normal vector (common perpendicular) to the gear tooth flank, $\mathscr{G}$ (to the pinion tooth flank, $\mathscr{P}$), that passes through the point of interest, m

Important to stress here that there is no need to verify whether or not the *first fundamental law of gearing is fulfilled* when the *second fundamental law of gearing* is fulfilled.

References

1. Radzevich, S.P., *Theory of Gearing: Kinematics, Geometry, and Synthesis*, 3rd Edition, revised and expanded, *CRC* Press, Boca Raton, FL, 2022, 1208 pages.

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3. Euler, L., "Supplementum. De Figura Dentium Rotarum". *Novi Commentarii adacemiae Petropolitanae* 11, 1767, pp. 207-231. (E330, *Opera omnia*, 17, pages 196-219). [1762/65; pp. 299–317].

4. Willis, R., *Principles of Mechanisms, Designed for the Use of Students in the Universities and for Engineering Students Generally*, London, John W. Parker, West Stand, Cambridge: J. & J.J. Deighton, 1841, 446 pages.

5. Pat. No. 407.437, (USA). *Machine for Planing Gear Teeth*./G.B. Grant, Filed: January 14, 1887 (serial No. 224,382), Patented: July 23, 1889.s.